

# The consistency relation in braneworld inflation

David Seery\* and Andy Taylor†

*Institute for Astronomy, Royal Observatory, Blackford Hill, Edinburgh, EH9 3HJ, UK*

(Dated: February 5, 2008)

The braneworld cosmology, in which our universe is imbedded as a hypersurface in a higher dimensional bulk, has the peculiar property that the inflationary consistency relation derived in a four-dimensional cosmology persists. This consistency condition relates the ratio of tensor and scalar perturbation amplitudes to the tensor spectral index produced during an epoch of slow-roll scalar field inflation. We attempt to clarify this surprising degeneracy. Our argument involves calculating the power spectrum of scalar field fluctuations around geometries perturbed away from the exact de Sitter case. This calculation is expected to be valid for perturbations which would not cause a late-time acceleration of the universe. We use these results to argue that the emergence of the same consistency relation in the braneworld can be connected with a specific property, that five-dimensional observables smoothly approach their four-dimensional counterparts as one takes the brane to infinite tension. We exhibit an explicit example where this does not occur, and in which a consistency relation does not persist.

PACS numbers: 98.80.Cq

astro-ph/0309512

## I. INTRODUCTION

Recent experimental results from the WMAP project [1] have lent strong support to the view that the observed homogeneity, isotropy and small-scale structure of the universe arises from an early period of accelerated expansion driven by a quantum field, the inflaton, that violates the weak energy condition. Although other possibilities exist [2, 3], the prototypical candidate for a matter component capable of supporting an inflationary epoch [4] of this type is a scalar field.

During inflation, all light fields (mass  $m < 3H/2$ , where  $H$  is the Hubble parameter during inflation) are excited and pick up a nearly scale invariant fluctuation whose characteristics are controlled by the inflaton potential. In the later universe, the inflaton fluctuation is communicated [5] to the curvature of spatial slices, seeding primordial structure formation, and observable today in the large scale distribution of the galaxies [6] and fluctuations in the Cosmic Microwave Background (CMB). Since the graviton is also massless one would expect small tensor perturbations to have been excited. Because of the extremely weak gravitational coupling, these perturbations would essentially not interact with other constituents of the universe on their journey towards us and could in principle be observable today, for example via its imprint in the polarisation field of the CMB, or with gravitational wave observatories coming on-line over the next decade. These tensor perturbations would almost certainly still be in their primordial state and could offer great insight into the early universe. In this case one would have four possible observables, the curvature amplitude,  $\Delta_\zeta^2$ , and spectral index,  $n_\zeta$ , and

the tensor amplitude,  $\Delta_T^2$ , and spectral index,  $n_T$ .

Since these quantities are all determined in the scalar field inflationary model by properties of the scalar potential, one expects to find some relation between them. In the context of standard cosmology, one finds [7]

$$\frac{\Delta_T^2}{\Delta_\zeta^2} \approx -8n_T, \quad (1)$$

to lowest order in the slow-roll approximation (see, eg., Ref. [8]). One can also calculate the next-order term in the slow-roll expansion [9], which does not preserve the functional form of Eq. (1).

Over and above the general current evidence in favour of an inflationary-like epoch, an observation of this relation in the real universe would provide extremely strong support for a minimal scalar field model. More complex models, such as those containing isocurvature modes, weaken this to an inequality, while observing an excess of primordial gravitational power would be a severe blow to the inflationary programme. However, we should stress that the exact non-perturbative relation between observables is not known; except in special cases, one does not know how to calculate away from the slow-roll approximation.

Over the last few years there has been considerable interest in cosmological models supporting large, extra dimensions [10, 11], motivated by developments in M-theory [12, 13]. It is therefore natural to ask both how inflation is implemented in these scenarios, and what possible modifications arise in its predictions for late-universe observables [14, 15]. Here one discovers a remarkable surprise. Although predictions for the tensor and scalar amplitudes and spectral indices are modified because they are sensitive to the behaviour of gravity in the large extra dimensions, the lowest-order consistency relation Eq. (1) survives [16, 17]. This is a non-trivial feature of the model, and at the time of writing we are not aware of any simple argument which demon-

\*Electronic address: djs@roe.ac.uk

†Electronic address: ant@roe.ac.uk

strates why this should be true. The unexpected appearance of the persistence of the consistency relation in the braneworld potentially jeopardizes the hope of observationally reconstructing the inflaton potential [18]. Hence an understanding of the origin of this degeneracy, and in particular deciding if it is universal, is essential to the inflaton potential programme.

In this paper, we clarify the circumstances under which one expects degeneracies between brane cosmology and conventional cosmology to persist. The consistency relation, Eq. (1), is derived in the brane and four-dimensional cases equally by calculating  $\Delta_\zeta^2$  and  $\Delta_T^2$  using quantum field theory for the idealization of a perfect de Sitter cosmology, where the Hubble parameter,  $H$ , is a constant. One then derives  $n_\zeta$  and  $n_T$  by supposing that  $H$  is changing extremely slowly, so that the approximation of a constant  $H$  over any few e-folds of inflation should be good. We will argue that the persistence of Eq. (1) can be regarded as a particular feature of this model: the tensor spectrum,  $\Delta_T^2$ , calculated in the brane world joins smoothly with the four-dimensional result as one decouples the brane from its surrounding environment. We verify this expectation by constructing an explicit example containing a small perturbation away from the exact de Sitter solution. This perturbation could be considered as a model of a bulk-brane interaction, or simply as an imbedded cosmology close to, but not exactly coinciding with, the de Sitter brane. One can successfully calculate the relevant amplitudes and spectral indices in this model, to first order in the perturbation. However, one finds that it is not possible to smoothly join the five-dimensional result to four-dimensional physics as one takes the brane to infinite tension. As a result, one does not recover a consistency relation.

This paper is organized as follows. In Section II we briefly review the calculation of the 4-D amplitudes and spectral indices and present a new derivation of the 5-D braneworld within the framework of Quantum Field Theory (QFT), both as a convenient reference for the later discussion and to establish our notation. In Section III we discuss the consistency relation in the four-dimensional case and in the braneworld. In Sections IV–V we calculate the effect of an arbitrary density perturbation  $\delta\rho(t)$  on the scalar field power spectrum in the four- and five-dimensional cases. We begin with an exact de Sitter cosmology fixed by  $H = \text{constant}$ , and introduce some small perturbation  $\delta H(t)$ . We assume it is still valid to treat the scalar field fluctuation  $\delta\phi$  as a free, massless field propagating on this background. The two-point function of  $\delta\phi$  can then be calculated both in the brane cosmology and the four-dimensional universe. In Section VI we apply the results obtained in Sections IV–V to study consistency relations in four and five dimensions. Our strategy is to write a relationship between observable parameters in the four dimensional case, and ask whether a comparable relationship holds in the braneworld. In fact it will turn out that the presence of a small perturbation prevents such a relationship. Finally, we state our con-

clusions (Section VII). Some material extraneous to the main text involving normalization of the graviton zero mode is presented for reference in an Appendix. We begin by reviewing the QFT calculation of the power spectra in four dimensions and presenting a new calculation in the 5-D braneworld.

## II. THE FOUR- AND FIVE-DIMENSIONAL LOWEST ORDER RESULTS

Scalar field inflation is based on free, massless field theory. If  $\phi$  is the inflaton with some potential  $V(\phi)$ , then one treats the gross evolution of  $\phi$  classically with the addition of some fluctuating part  $\delta\phi$  which is to be treated quantum mechanically. It is a good approximation to take  $\delta\phi$  to be a free, massless field. The inflaton field  $\phi$  itself will not enter into our considerations, so for the remainder of this paper we simply drop the  $\delta$  from the fluctuating field  $\delta\phi$ .

### A. 4-d scalar power spectrum

Accordingly, let  $\phi$  be a free, massless scalar field. Its correlation functions are controlled by the functional integral,

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int [\mathcal{D}\phi] \phi(x_1) \cdots \phi(x_n) \exp \left( -\frac{i}{2} \int_{\mathcal{M}} dx \phi \square \phi \right), \quad (2)$$

where  $\mathcal{M}$  is the background spacetime with metric  $g_{\nu\mu}$  and invariant volume measure  $dx$ , and we have chosen units in which  $\hbar = 1$ . The operator  $\square = \nabla^\mu \nabla_\mu$ , where  $\nabla_\mu$  is the covariant derivative compatible with  $g_{\nu\mu}$ . One can evaluate the functional integral in Eq. (2) explicitly, for example for the two-point function. In this case Wick's theorem shows that  $\langle \phi(x_1) \phi(x_2) \rangle = -i \square^{-1}(x_1, x_2)$ .

Now let  $\mathcal{M}$  be de Sitter space. We choose local coordinates in which the metric takes the form

$$ds^2 = \frac{1}{H^2 \tau^2} (-d\tau^2 + \delta_{ij} dx^i dx^j). \quad (3)$$

There are three Killing vectors  $\partial/\partial x^i$  which act transitively on slices  $\tau = \text{constant}$ . The points  $x_1$  and  $x_2$  have coordinates  $x_1 = (\tau_1, \mathbf{x}_1)$ ,  $x_2 = (\tau_2, \mathbf{x}_2)$ . Here,  $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$  but one can choose the range of  $\tau$ . To cover the entire manifold, one takes  $\tau \in \mathbb{R}$ . The spatial slices in this case are  $S^3$ . Alternatively [19], one can work on the half space  $\tau \in \mathbb{R}^- \equiv (-\infty, 0]$  which corresponds to the portion of the manifold covered by the cosmic-time form of the metric with flat spatial slices,  $ds^2 = -dt^2 + e^{2Ht} \delta_{ij} dx^i dx^j$ . This choice is particularly common and convenient when discussing inflation. In either case, the infinite past corresponds to  $\tau \rightarrow -\infty$ . We work on the full space  $\tau \in \mathbb{R}$  because it is easier to get

the boundary conditions right, but in any case  $\tau$  usually appears in the combination  $k\tau$  with some wavenumber  $k$ , and we will take the large scale limit  $k \rightarrow 0$ . In this case, one finds that if  $\tau_1 > \tau_2$  the propagator satisfies (see, eg., [20])

$$\langle \phi(x_1)\phi(x_2) \rangle = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{H^2 \tau_1 \tau_2 \pi}{4k} L^{(1)}(-k\tau_1) \times L^{(2)}(-k\tau_2) e^{-i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)}, \quad (4)$$

where

$$L^{(1,2)}(z) = (z)^n H_{3/2}^{(n)}(z) \quad \text{for } n = 1, 2. \quad (5)$$

The boundary conditions here are chosen to correspond with the adiabatic (Bunch-Davies) vacuum prescription [20, p. 132], that Eq. (4) is close to the flat space limit,  $\sim e^{-ik|\tau|}$ , whenever the wavevector is small ( $k \rightarrow \infty$ ) compared to the curvature of spacetime or when approaching the asymptotically early or late times,  $\tau \rightarrow \pm\infty$ . Since  $\phi$  is free, there are no singularities requiring renormalization in the operator product expansion, although the propagator is log divergent in both the ultra-violet and infra-red. One can take the  $\mathbf{x}_1 \rightarrow \mathbf{x}_2$  limit to find an effective variance,  $\sigma_\phi^2(x) = \langle \phi(x)\phi(x) \rangle$ , which satisfies

$$\sigma_\phi^2(\tau) = (H\tau)^2 \int_0^\infty \frac{k dk}{8\pi} L^{(1)}(-k\tau) L^{(2)}(-k\tau), \quad (6)$$

where  $\tau_1 = \tau_2 = \tau$ . This is independent of the spatial coordinates  $\mathbf{x}$ .

Throughout this paper the power spectrum,  $\Delta_X^2$ , of some field,  $X$ , is defined as

$$\Delta_X^2(k) = \frac{d\sigma_X^2(k)}{d \ln k}, \quad (7)$$

where  $\sigma_X^2$  is the variance in  $X$ . The power spectrum  $\Delta_\phi^2$  on large scales,  $k \rightarrow 0$ , approaches the well-known finite limit

$$\Delta_\phi^2 = (H/2\pi)^2. \quad (8)$$

The intrinsic curvature perturbation [5] induced by this fluctuation in the scalar field satisfies  $\zeta = H\delta\phi/\dot{\phi}$ , where  $\dot{\phi}$  is the classical background evolution. Our conventions for this quantity coincide with Ref. [21]. Therefore,

$$\Delta_\zeta^2 = (H^2/\dot{\phi}^2)\Delta_\phi^2. \quad (9)$$

One can now define a sequence of observables based on logarithmic derivatives of  $\Delta_X^2$  with respect to  $\log k$ . The two lowest members of this sequence are the spectral index,  $n_X$ , and its running,  $r_X$ :

$$n_X = \frac{d \log \Delta_X^2}{d \log k} \quad \text{and} \quad r_X = \frac{d^2 \log \Delta_X^2}{d[\log k]^2}. \quad (10)$$

## B. 4-d tensor power spectrum

Linear gravitational waves consist of small perturbations  $E_{ab}$  to the metric:

$$ds^2 = (\eta_{ab} + E_{ab}) dx^a dx^b. \quad (11)$$

As a representation of the Lorentz group, the metric perturbation  $E_{ab}$  is reducible [22]. To pick out the pure spin-2 contribution one imposes the condition that  $E_{ab}$  be transverse ( $\nabla^a E_{ab} = 0$ ) and traceless ( $\text{Tr } E_{ab} = 0$ ) with respect to the Lorentz metric  $\eta_{ab}$ .

The gravitational action is

$$S = -\frac{1}{2\kappa_4^2} \int dx R, \quad (12)$$

where  $R = \text{Tr } R_{ab}$  is the trace of the Ricci tensor. To find an action for  $E_{ab}$  one expands  $R$  to second order in  $E_{ab}$ . The result is

$$S_2 = -\frac{1}{8\kappa_4^2} \int dx E^{ab} \square E_{ab}, \quad (13)$$

where  $\square = \nabla^a \nabla_a$  is still the scalar d'Alembertian. Thus, the Lorentz indices  $(a, b)$  of  $E_{ab}$  are not noticed and effectively label an internal  $SO(3, 1)$ , so that  $E_{ab}$  behaves like some number of fields in the trivial (scalar) representation of the Lorentz group. One includes one such field for each polarization state of the graviton. In four dimensions, there are two such polarizations; it is conventional to label one polarization by  $+$  and the other by  $\times$ . This means that the action for gravitational perturbations is the same as two copies of the scalar action, Eq. (2), except that the overall normalization is changed by a factor  $(4\kappa_4^2)^{-1}$ . As a result, the gravitational power spectrum satisfies

$$\Delta_T^2 = 8\kappa_4^2 \Delta_\phi^2 = 2\kappa_4^2 \frac{H^2}{\pi^2}. \quad (14)$$

## C. 5-d braneworld

In the braneworld, one works on an anti-de Sitter or Schwarzschild-anti de Sitter (SAdS) [10, 23, 24, 25, 26] manifold  $\mathcal{M}$  with an imbedded hypersurface  $\Sigma$ . Throughout this paper, we consider only the pure anti-de Sitter case. The hypersurface  $\Sigma$  supports the various matter and gauge fields which comprise our cosmology. The metric is taken to be [27, 28]

$$ds^2 = -n^2(t, y) dt^2 + a^2(t, y) \delta_{ij} dx^i dx^j + dy^2. \quad (15)$$

One can replace  $\delta_{ij}$  by the metric  $\gamma_{ij}$  of any maximally symmetric three-geometry. The brane is considered to be imbedded at  $y = 0$  with a  $\mathbb{Z}_2$  symmetry, and the metric functions  $a(t, y)$  and  $n(t, y)$  depend on the four-dimensional brane geometry. For this reason, Eq. (15) is

not a product manifold; this is sometimes expressed by saying that  $\mathcal{M}$  is a warped compactification.

The  $\mathbb{Z}_2$  symmetry acts by  $y \mapsto -y$  and is motivated from heterotic M-theory [12, 13]. We loosely refer to this construction as an orbifold; one can choose either to work on the full orbifold, or the line interval corresponding to just one half. In general we will work on the  $y > 0$  branch rather than on the full space; this makes no difference to computations, except that factors of 2 must occasionally be added by hand.

Except in the case that the brane is empty [10], these coordinates cover only a patch of AdS [24], so the coordinate  $y$  does not take on unboundedly large values but instead only assumes values in some interval  $y \in [-y_h, y_h]$ . In the  $y > 0$  picture, we restrict  $y$  to  $y \in [0, y_h]$ . The location of the coordinate horizon at  $y = y_h$  depends on the brane tension and matter theory [15].

The effective Einstein equations on the brane were found in [29]. They are, with  $\kappa_4^2$  and  $\kappa_5^2$  the four- and five-dimensional gravitational couplings respectively,

$$G_{ab} = \kappa_4^2 T_{ab} + \kappa_5^2 \pi_{ab} + E_{ab}, \quad (16)$$

where  $G_{ab}$  is the effective four-dimensional Einstein tensor,  $T_{ab}$  is the energy-momentum tensor of whatever matter and gauge degrees of the freedom reside on the brane,  $\pi_{ab}$  is quadratic in  $T$  and  $E$  is the limit as one approaches the brane of the ‘electric’ part of the Weyl tensor in the bulk. The hierarchy of Planck scales is controlled by a parameter  $\mu$ ,

$$\mu = \frac{\kappa_4^2}{\kappa_5^2} = \frac{M_5^3}{M_4^2}. \quad (17)$$

The Friedmann equation receives quadratic corrections from the term  $\pi_{ab}$ , involving the inverse brane tension  $\lambda$  [28],

$$H^2 = \frac{\kappa_4^2}{3} \rho \left( 1 + \frac{\rho}{2\lambda} \right). \quad (18)$$

If there is no four-dimensional cosmological constant, then the hierarchy parameter  $\mu$  is the AdS curvature scale and is related to  $\lambda$  by

$$\lambda = \frac{6\mu}{\kappa_5^2}. \quad (19)$$

For future use, we note that the  $\lambda \rightarrow 0$  limit can be identified with  $\mu \rightarrow 0$  at fixed  $\kappa_5$ , and similarly as  $\lambda \rightarrow \infty$ . The  $\lambda \rightarrow 0$  limit sends  $\kappa_4$  to zero and so switches off four-dimensional gravity on the brane.

The explicit form of  $a(t, y)$  and  $n(t, y)$  is, for  $a$ , [28]

$$\left( \frac{a}{a_b} \right)^2 = \frac{H^2}{2\mu^2} [\cosh 2\mu(y_h - y) - 1], \quad (20)$$

and the conventional choice of  $n = \dot{a}/\dot{a}_b$ ,

$$\frac{a_b}{a} n = 1 - \left( \frac{a_b}{a} \right)^2 \frac{\dot{H}}{2\mu^2} \left( \frac{H^2}{\mu^2 + H^2} \cosh 2\mu(y_\infty - y) - 1 \right) \quad (21)$$

and  $a_b$  refers to the scale factor on the brane,  $a(t, y = 0)$ . The quantities  $y_h(t)$  and  $y_\infty(t)$  are defined by, which are functions of  $t$ ,

$$\begin{aligned} \tanh 2\mu y_h &= \frac{(1 + H^2/\mu^2)^{1/2}}{1 + H^2/2\mu^2}, \\ \tanh 2\mu y_\infty &= \frac{1}{(1 + H^2/\mu^2)^{1/2}}. \end{aligned} \quad (22)$$

Clearly, from Eq. (20),  $y = y_h$  is always a zero of  $a$  and defines a Cauchy horizon or coordinate singularity, where the Gaussian normal coordinates used to write Eq. (15) break down. There is an analytic extension beyond this horizon [24]. The location  $y = y_h$  is a global minimum for  $a$ . Although  $a$  always goes to zero at the Cauchy horizon, in general  $n$  does not; indeed, it is typically discontinuous there. However the values of  $a$  and  $n$  for  $y > y_h$  are not meaningful, so this discontinuity is not seen by observers in the spacetime. There is no simple geometric interpretation for  $y_\infty$ .

#### D. 5-d scalar power spectrum

Let  $\phi$  be a free massless scalar field propagating over  $\Sigma$ . Then the propagator for  $\phi$  is still defined by Eq. (2) (with integration over spacetime  $\mathcal{M}$  replaced by integration over  $\Sigma$ ) and is exactly the same as the four-dimensional case, Eq. (9) [21].

#### E. 5-d tensor power spectrum

The situation for gravitational perturbations is more complicated, and was first analysed by [15] (see also Refs. [30, 31, 32]) in the Schrödinger picture. Here we repeat the calculation in a QFT. We will use this approach to generalize the calculation to an arbitrarily perturbed de Sitter brane in the Section V.

Let  $E_{ij}$  be a small perturbation of the metric Eq. (15):

$$ds^2 = -n^2(t, y) dt^2 + a^2(t, y)(\delta_{ij} + E_{ij}) dx^i dx^j + dy^2 \quad (23)$$

where  $E_{ij}$  is transverse and traceless with respect to the three-dimensional spatial metric  $\delta_{ij}$ . Just as in four dimensions, for the purposes of the resulting field theory, the indices  $(i, j)$  act as internal  $SO(3)$  indices and  $E$  behaves like two copies of a field in the trivial (scalar) representation of  $SO(3)$ . Just as in any Kaluza-Klein type decomposition, in order to make up the full  $SO(3, 1)$  graviton, one should include contributions from a graviscalar  $\phi$  and graviphoton  $A_i$  which are the other components of a decomposition of perturbations to the metric under the isometry group of Eq. (15). We ignore  $\phi$  and  $A_i$ ; they can be set to zero by a gauge transformation and do not contribute to the vacuum fluctuation during inflation [33].

The two-point function for  $E_{ij}$  satisfies

$$\begin{aligned} \langle E^{ij}(x_1)E_{ij}(x_2) \rangle &= \int [\mathcal{D}E_{mn}] E^{ij}(x_1)E_{ij}(x_2) \\ &\times \exp \left[ -\frac{i}{8\kappa_5^2} \int_{\mathcal{M}} dx E^{mn} \left( \frac{\square_{\text{KG}}}{n^2} + \square_Y \right) E_{mn} \right], \end{aligned} \quad (24)$$

where we have decomposed the 5-D braneworld d'Alembertian,  $\square_{\text{BW}}$ , into two terms  $\square_{\text{KG}}$  and  $\square_Y$ , defined by

$$\begin{aligned} \square_{\text{KG}} &= -\frac{\partial^2}{\partial t^2} - \left( 3\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \frac{\partial}{\partial t} + \frac{n^2}{a^2} \Delta, \\ \square_Y &= \frac{\partial^2}{\partial y^2} + \left( 3\frac{a'}{a} + \frac{n'}{n} \right) \frac{\partial}{\partial y}. \end{aligned} \quad (25)$$

Because Eq. (15) is not a product metric,  $\square_{\text{KG}}$  and  $\square_Y$  are not the on- and off-brane d'Alembertians.  $\square_{\text{KG}}$  is similar to the Klein–Gordon operator on slices  $y = \text{constant}$ , but carries an overall  $y$ -dependence owing to the fact that both  $n$  and  $a$  are in general functions of  $y$ . Similarly,  $\square_Y$  depends in general on both  $t$  and  $y$ . However, in the important special case that the brane is endowed with a de Sitter geometry  $\dot{H} = 0$ , then these operators separate [15]. In this case  $\square_Y$  is an honest Sturm–Liouville operator and one can write  $E_{ij}$  as a sum over its harmonics. This also allows us to re-express the path integral measure as a product of four-dimensional path integrals, and we make contact with the four-dimensional physics using this device.

Following [15] we define a set of weighted eigenfunctions,  $\mathcal{E}_\alpha(y)$ , of  $\square_Y$  by

$$\square_Y \mathcal{E}_\alpha(y) = -(\alpha^2/n^2) \mathcal{E}_\alpha(y). \quad (26)$$

In Sturm–Liouville form this says  $(n^4 \mathcal{E}'_\alpha)' + \alpha^2 n^2 \mathcal{E}_\alpha = 0$ . The  $\mathcal{E}_\alpha$  can be chosen to be orthonormal in the standard inner product defined by this measure, ie.,

$$(\mathcal{E}_\alpha | \mathcal{E}_\beta) = 2 \int_0^{y_h} d\mu(y) \mathcal{E}_\alpha^* \mathcal{E}_\beta = \delta_{\alpha\beta}, \quad (27)$$

where  $d\mu(y) = n^2 dy$ , provided the  $\mathcal{E}_\alpha$  obey suitable boundary conditions at  $y = 0$  and  $y = y_h$ ; typically such boundary conditions are the Dirichlet–Neumann conditions  $m\mathcal{E}'_\alpha + n\mathcal{E}_\alpha = 0$ , where  $m$  and  $n$  are independent of  $\alpha$ . We have added a factor 2 by hand in the above equation, to take account of the other branch of the orbifold. For physical reasons [15, 30] we choose the derivatives of the  $\mathcal{E}_\alpha$  to vanish at  $y = 0$  and  $y = y_h$ . The allowed values of  $\alpha$  consist of a discrete zero-mode bound state at  $\alpha = 0$  and a continuum of massive modes for  $\alpha > 3H/2$ . We now write  $E_{ij}$  as an eigenfunction decomposition,

$$E_{ij}(\tau, \mathbf{x}, y) = \sum_{\alpha, \omega} E_\omega^\alpha(\tau, \mathbf{x}) \mathcal{E}_\alpha(y) \varepsilon_{ij}^\omega, \quad (28)$$

where the scalar  $E_\omega^\alpha(\tau, \mathbf{x})$  is independent of  $y$ , and  $\varepsilon_{ij}^\omega$  is a constant  $SO(3)$  polarization tensor labeled by an index

$\omega$ . To take advantage of this decomposition we rewrite the path integral measure as

$$[\mathcal{D}E_{ij}] \propto \prod_{\alpha, \omega} [\mathcal{D}E_\omega^\alpha], \quad (29)$$

and the two-point function becomes

$$\begin{aligned} \langle E^{ij}(x_1, y_1)E_{ij}(x_2, y_2) \rangle &= \\ &\int \prod_{\alpha, \omega} [\mathcal{D}E_\omega^\alpha] \left( \sum_{\alpha', \alpha'', \omega'} E_{\omega'}^{\alpha'} E_{\omega'}^{\alpha''} \mathcal{E}_{\alpha'}(y_1) \mathcal{E}_{\alpha''}(y_2) \right) \\ &\times \exp \left( -\frac{i}{8\kappa_5^2} \int_{\Sigma} dx \sum_{\omega} E_\omega^\alpha (\square_{\text{KG}} - \alpha^2) E_\omega^\alpha \right). \end{aligned} \quad (30)$$

We have integrated over the transverse dimension, so all factors of  $n$  have disappeared from the measure and  $\square_{\text{KG}}$ ; cf. Eq. (27). Thus the field  $E_{ij}$  behaves like a collection of four-dimensional Klein–Gordon fields with masses described by the allowed values of  $\alpha$ . At low energies, or during inflation, only the  $\alpha = 0$  zero-mode will be excited, so since  $\mathcal{E}_0$  is independent of  $y$ , one has

$$\begin{aligned} \langle E^{ij}(x_1)E_{ij}(x_2) \rangle &\approx \mathcal{E}_0^2 \sum_{\omega'} \int [\mathcal{D}E_\omega^0] E_{\omega'}^0 E_{\omega'}^0 \\ &\times \exp \left( -\frac{i}{8\kappa_5^2} \int_{\Sigma} dx \sum_{\omega} E_\omega^0 \square_{\text{KG}} E_\omega^0 \right) \\ &= -8i\mathcal{E}_0^2 \kappa_5^2 \square_{\text{KG}}^{-1}(x_1, x_2). \end{aligned} \quad (31)$$

This is still a free theory, so there is no obstruction to taking the limit  $x_1 \rightarrow x_2$  and the power spectrum follows in the same way as the four-dimensional case. When the imbedded geometry is purely de Sitter  $n$  is independent of  $t$ , and  $a = e^{Ht}n$ , so  $\square_{\text{KG}}$  is the same operator as  $\square$  on 4-D de Sitter space. Re-using the result Eq. (6), one obtains

$$\Delta_T^2 = 8\mathcal{E}_0^2 \kappa_5^2 \Delta_\phi^2 = 2\mathcal{E}_0^2 \kappa_5^2 \frac{H^2}{\pi^2}. \quad (32)$$

The polarizations are labeled by  $\omega \in \{+, \times\}$  as in four dimensions. Because the  $\mathcal{E}_0$  are just constants, this is no more than a renormalization of the four-dimensional power spectrum by some function of  $H$  and  $\mu$ . In order to make this property more transparent, it is conventional to re-write the power spectrum in terms of the effective four-dimensional gravitational coupling  $\kappa_4^2$  by setting  $\mathcal{E}_0^2 = \mu F^2$ , where  $F$  is a constant to be fixed by the normalisation of  $\mathcal{E}_0$  (see Appendix). With these conventions, one has the simple result,

$$\Delta_T^2 = 2\kappa_4^2 F^2 \frac{H^2}{\pi^2}. \quad (33)$$

### III. THE CONSISTENCY RELATION

We now briefly describe how the consistency relation, Eq. (1), arises in four dimensions, and in the braneworld.

Consider four-dimensional inflation driven by some scalar field. The matter and tensor power spectra satisfy Eqs. (9) and (14). Therefore, their ratio is independent of  $\Delta_\phi^2$ :

$$\frac{\Delta_T^2}{\Delta_\zeta^2} = 8\kappa_4^2 \frac{\dot{\phi}^2}{H^2}. \quad (34)$$

This arises because  $\phi$  and  $E_{ab}$  share the same action, and the result is that the ratio Eq. (34) depends only on the relation between  $H$  and  $\phi$ , and not on the details of the quantum field theory calculation leading to  $\Delta_\phi^2$ . If one assumes that the scalar field  $\phi$  is the only constituent of the universe, then  $H$  evolves with  $\phi$  according to the equation

$$H' = -\frac{\kappa_4^2}{2}\dot{\phi}. \quad (35)$$

A prime denotes a derivative with respect to  $\phi$ . This is often called the Hamilton–Jacobi equation [9]. One then eliminates  $\dot{\phi}$  from the ratio Eq. (34) to find

$$\frac{\Delta_T^2}{\Delta_\zeta^2} = \frac{32}{\kappa_4^2} \left( \frac{H'}{H} \right)^2. \quad (36)$$

We define a tensor spectral index  $n_T$  as in Eq. (10). To evaluate this we endow  $H$  with some extremely slow time dependence owing to motion of  $\phi$  on very long time scales. One quantifies this time dependence by introducing slow roll parameters  $\varepsilon$ ,  $\eta$  [8] defined by

$$\varepsilon = \frac{2}{\kappa_4^2} \left( \frac{H'}{H} \right)^2 \quad \text{and} \quad \eta = \frac{2}{\kappa_4^2} \frac{H''}{H} \quad (37)$$

and demanding that these are both small,  $\varepsilon, \eta \ll 1$ . One typically works to first order in  $\max\{\varepsilon, \eta\}$ .

On horizon scales  $k \approx aH$ , the differential  $d \log k$  is effectively  $d \log a$  since we are assuming that change in  $H$  is negligible in comparison with change in  $a$ . This approximation is expected to be good because  $a = e^{Ht}$  is moving exponentially fast in comparison with  $H$ , which assumed almost constant. We use this result to evaluate  $n_T$ .

With these choices, one obtains Eq. (1). This result depends on the relationship between  $\dot{\phi}$  and  $H$  and the functional form of  $\Delta_\phi^2$ . In particular, if  $\Delta_\phi^2$  is just monomial in  $H$ , ie.,  $\Delta_\phi^2 \propto H^n$  for some  $n$ , and  $\dot{\phi}$  is linear in  $H'$ , then one will obtain a consistency relation of this type, with coefficient depending on the details of the relationships.

### A. The consistency relation in the braneworld

In the braneworld, the ratio  $\Delta_T^2/\Delta_\zeta^2$  is still independent of  $\Delta_\phi^2$ . One can again write a relation between  $\dot{\phi}$  and  $H$ ,

which is the analogue of the four-dimensional Hamilton–Jacobi equation. This is modified by the anti-de Sitter radius  $\mu$ , and becomes

$$\dot{\phi} = -\frac{2\mu}{\kappa_4^2} \frac{H'}{(H^2 + \mu^2)^{1/2}}. \quad (38)$$

This means that the ratio of the amplitudes satisfies

$$\frac{\Delta_T^2}{\Delta_\zeta^2} = \frac{32\mu^2}{\kappa_4^2} F^2 \left( \frac{H'}{H} \right)^2 \frac{1}{H^2 + \mu^2}. \quad (39)$$

The tensor spectral index no longer depends merely on the functional form of  $\Delta_\phi^2$ , which is unchanged from the four-dimensional case, but instead receives non-trivial corrections from the renormalization  $F^2$ . It satisfies  $d \log \Delta_T^2 = 2d \log HF$ . However,  $HF$  satisfies a particular differential equation [16]:

$$d \log HF = \frac{\mu F^2}{(H^2 + \mu^2)^{1/2}} d \log H. \quad (40)$$

Combining  $n_T$  with Eq. (38) and Eq. (40) gives back the consistency relation Eq. (1). The relation Eq. (40) is derived, together with an explicit expression for  $F$ , in Appendix A.

Although we have derived this result only for the case of a pure anti-de Sitter bulk with  $\mathbb{Z}_2$  symmetry, the appearance of the consistency relation holds rather more generally. In particular, Huey & Lidsey [16] have argued that it persists if one allows different anti-de Sitter curvatures  $\mu_<$ ,  $\mu_>$  on the  $y < 0$  and  $y > 0$  branches.

### B. The large $\lambda$ limit

This coincidence appears remarkable. We have non-trivially modified gravity in the brane world scenario, but when one asks about relationships between observable parameters, the modification becomes invisible. For a deeper understanding of brane cosmology in general and to clarify the observational position, one would like to understand how this circumstance arises. In particular, one would like to know whether its appearance in this model is in some sense accidental, or is enforced by deeper principles. Our claim is that its appearance is only accidental.

The appearance of the consistency relation Eq. (1) in the brane universe theory depends on the differential equation Eq. (40), for which it is difficult to find any clear geometrical or physical interpretation. Its effect is to render the relationship between  $\Delta_\zeta^2$ ,  $\Delta_T^2$  and  $n_T$  independent of the brane tension  $\lambda$  which appears in the Friedmann equation. One approach to the problem is to consider the limit of very large  $\lambda$ , where the brane is close to decoupling from the bulk and one is nearly in the regime of four-dimensional cosmology. Physics on the brane is supposed to be insensitive to the value of  $\lambda$ , which we formalize by demanding that the energy–momentum tensor

of matter carried by the brane is independent of  $\lambda$ . If this is true, then the density  $\rho$  does not depend on  $\lambda$  either and the only appearance of the tension is in the Friedmann equation, as explicitly written in Eq. (18).

In fact this is part of a more general correspondence principle, that one should recover four dimensional physics as the brane becomes infinitely stiff and decouples from the bulk. This is true for purely four-dimensional quantities, such as observables in any quantum field theory fixed to the brane, but it need not be true for gravitational quantities. Consider, for example, the effective Einstein equations Eq. (16). As  $\lambda \rightarrow \infty$ , the quadratic term  $\pi$  which leads to the quadratic  $\rho$  dependence in Eq. (18) decouples, but  $E_{ab}$  need not. This means that unless  $E_{ab} = 0$ , then as one decouples the brane one need not see gravitational quantities such as  $\Delta_T^2$  and  $n_T$  computed using Eq. (16) approach their counterparts computed in four dimensional Einstein gravity.

For this reason, a useful way to look for models which might break consistency relation-type degeneracies between brane and four-dimensional cosmologies is to search for examples in which observable quantities do not smoothly approach their four-dimensional Einstein counterparts as one decouples the brane. In the next section, we will examine such a model.

To decide how the limit  $\lambda \rightarrow \infty$  ought to be taken, one needs some sort of prescription for handling the matter on  $\Sigma$ . For example, the Einstein–Klein–Gordon system for a four-dimensional Robertson–Walker universe or an imbedded Robertson–Walker universe in the five-dimensional case can be written as a pair of equations for the scalar field,

$$\rho = \dot{\phi}^2/2 + V, \quad \text{and} \quad \rho' = -3H\dot{\phi}, \quad (41)$$

supplemented with an equation for the Hubble parameter, Eq. (18) in the 5d brane case or  $H^2 = \kappa_4^2 \rho/3$  in the 4d case. Once one specifies any one of these quantities in terms of  $\phi$ , the other two are determined by direct differentiation or quadrature, as appropriate.

de Sitter space corresponds to a fixed value of  $H$ . In four dimensions, this translates straightforwardly to a fixed  $\rho$  and a fixed  $V$ . In five dimensions, the situation is not quite so simple; in particular, as the tension  $\lambda$  varies, one must vary  $\rho$  and  $V$  to keep  $H$  fixed. Therefore, if one wishes to deal with some given four-dimensional cosmology and then imbed this as a brane universe, one must specify which characteristic, selected from  $H$ ,  $\rho$  or  $V$ , is to be used to determine the brane geometry. The other two will vary with  $\lambda$ . In the present case, we have a single fixed  $H$ . If one now applies a perturbation  $\delta H(t)$ , then we fix our notion of what is the same cosmology imbedded on a brane by asking that one obtains the same perturbation  $\delta H(t)$ . This must be sourced by a perturbation  $\delta\rho(t, \lambda)$  which is also a function of  $\lambda$ .

#### IV. FLUCTUATIONS IN A PERTURBED 4-D DE SITTER SPACE

In this section, we aim to calculate the power spectrum of a scalar field propagating over a background de Sitter cosmology with some first order perturbation. Let  $M$  be four-dimensional de Sitter space with Hubble parameter  $H_0$ . Consider a matter density perturbation  $\delta\rho$  of square zero which induces a change  $\delta H$  in the Hubble parameter. We wish to calculate the power spectrum of a free, massless scalar field propagating on this geometry.

The functional integral Eq. (2) defining correlation functions in the scalar field theory is unchanged, so the two point function is still given by  $-i\Box^{-1}$ , although  $\Box$  now includes contributions of  $O(\delta H)$  from the perturbation. For convenience we adopt the convention of writing the inverse d'Alembertian as a Green's function;  $G = \Box^{-1}$ . Then  $G$  satisfies

$$\Box G(x_1, x_2) = \delta(x_1 - x_2) \quad (42)$$

where  $\delta(x_1 - x_2)$  is the covariant Dirac delta function, which can be written in terms of the coordinates  $\mathbf{x}_1^i$ ,  $\mathbf{x}_2^i$  of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  as  $(-\det g)^{-1/2} \prod_i \delta(\mathbf{x}_1^i - \mathbf{x}_2^i)$ . Although the Green's function is symmetric between  $\mathbf{x}_1$  and  $\mathbf{x}_2$  it is helpful in calculations to adopt the convention that  $G$  is a function of one set of coordinates, for which we choose the  $\mathbf{x}^i$ . The coordinates  $\mathbf{x}^i$  are then considered to be constants.  $G$  can be solved as a perturbation expansion in  $\delta H$ . We write  $G = G_0 + \delta G$ , and because there still exist the three spacelike Killing vectors  $\partial/\partial x^i$  it is useful to diagonalize  $G$  by writing it as a Fourier transform,  $G(\mathbf{x}, \mathbf{x}_2) = \int d^3k (2\pi)^{-3} G(\mathbf{k}; t, t_2) e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_2)}$ . The  $O(1)$  equation for  $G_0(\mathbf{k})$  is

$$\left( \frac{\partial^2}{\partial t^2} + 3H_0 \frac{\partial}{\partial t} + \frac{k^2}{a^2} \right) G_0 = \frac{\delta(t - t_2)}{a^3}. \quad (43)$$

Here the independent variable  $t$  on the left hand side is taken to be the  $t$  coordinate of  $x_1$ , as discussed above. We still write the  $t$  coordinate of  $x_2$  as  $t_2$ . By making the transition to conformal time  $\tau$  defined by  $dt = a d\tau$  and rescaling  $G_0 \rightarrow aG_0$ , this becomes

$$\mathcal{B}_{3/2} G_0 = \frac{\delta(\tau - \tau_2)}{a^3}. \quad (44)$$

The operator  $\mathcal{B}_\mu$  on the left-hand side will occur again, so it is convenient to have a notation for it. It is defined by

$$\mathcal{B}_\mu = \frac{\partial^2}{\partial \tau^2} + \left( k^2 - \frac{\mu^2 - 1/4}{\tau^2} \right). \quad (45)$$

The function  $G_0$  is almost everywhere zero and so should lie in the space  $\ker \mathcal{B}_\mu$  of functions annihilated by  $\mathcal{B}_\mu$ . This is a two-parameter space of functions spanned by linear combinations of the form

$$\Upsilon(-k\tau)^{1/2} H_\mu^{(1)}(-k\tau) + \Xi(-k\tau)^{1/2} H_\mu^{(2)}(-k\tau) \quad (46)$$

for some constants  $\Upsilon$  and  $\Xi$ . The precise linear combination one chooses for  $G_0$  depends on the boundary conditions in the far past ( $\tau \rightarrow -\infty$ ) and the far future ( $\tau \rightarrow +\infty$ ). A common choice is the Bunch–Davies vacuum [20], where one chooses  $G_0$  to behave like the Hankel function  $H^{(2)}$  near  $\tau \rightarrow +\infty$ , and like  $H^{(1)}$  near  $\tau \rightarrow -\infty$ . Demanding that  $G_0$  be continuous at  $\tau = \tau_q$ , but with a step in derivative to satisfy Eq. (44) gives back the four dimensional propagator quoted as Eq. (4).

The  $O(\delta H)$  equation is

$$\left( \frac{\partial^2}{\partial t^2} + 3H_0 \frac{\partial}{\partial t} + \frac{k^2}{a^2} \right) \delta G = \left( \frac{2k^2}{a^2} \frac{\delta a}{a} - 3\delta H \frac{\partial}{\partial t} \right) aG_0 - 3 \frac{\delta a}{a} \frac{\delta(t - t_2)}{a^3}. \quad (47)$$

One now follows the same procedure as above, writing  $\delta G \rightarrow a\delta G$  and changing to conformal time. The result is

$$\mathcal{B}_{3/2} \delta G = 2 \left[ k^2 G_0 \frac{\delta a}{a} - 3\delta H \left( a \frac{\partial G_0}{\partial \tau} - a^2 H_0 G_0 \right) \right] - 3 \frac{\delta a}{a} \frac{\delta(\tau - \tau_2)}{a} \quad (48)$$

$$= \mathcal{J}(\tau, \tau_2) - 3 \frac{\delta a}{a} \frac{\delta(\tau - \tau_2)}{a}. \quad (49)$$

We have written the non-distributional part of the source term, in square brackets on the right-hand side, as  $\mathcal{J}(\tau, \tau_2)$ . This is a useful abbreviation, but in any case it is convenient to work with a quite general source term because we will be able to reuse the result in §V below.

$\mathcal{B}_\mu$  is a well-defined Sturm–Liouville operator for any  $\mu$  except at  $\tau = 0$ . We define a set of eigenfunctions  $\{\phi_m\}$  of  $\mathcal{B}_{3/2}$  by

$$\mathcal{B}_{3/2} \phi_m = -m^2 \phi_m, \quad (50)$$

which is supplemented by appropriate boundary conditions. We take the  $\{\phi_m\}$  to be defined on (at least)  $\tau \in \mathbb{R}^-$ , so the boundary condition at  $\tau = -\infty$  is expected to be immaterial, provided the  $\{\phi_m\}$  decay sufficient fast there. At  $\tau = 0$ , we demand that the  $\{\phi_m\}$  be regular. Then a standard argument shows that  $\mathcal{B}_{3/2}$  is self-adjoint on the  $\{\phi_m\}$  and consequently that we may choose the  $\{\phi_m\}$  to be orthonormal for different  $m$ . Explicitly, the appropriately normalized  $\{\phi_m\}$  satisfy

$$\phi_m(k, \tau) = (-\sqrt{k^2 + m^2} \tau)^{1/2} J_{3/2}(-\sqrt{k^2 + m^2} \tau), \quad (51)$$

the Bessel function  $J_{3/2}$  being chosen to keep the  $\phi_m$  regular at  $\tau = 0$ .

Our strategy is to solve Eq. (49) by taking a transform in the  $\{\phi_m\}$ . This can be thought of as the continuum limit [34] of an eigenseries expansion in the  $\{\phi_m\}$ , cut off at some limiting value  $\tau = -\tau_{\text{limit}}$ , as  $\tau_{\text{limit}} \rightarrow \infty$ . One writes the term  $\mathcal{J}$  as a  $\mathcal{B}_{3/2}$ -transform,

$$\mathcal{J}(k, \tau, \tau_2) = \int_{-\infty}^{\infty} dm \phi_m(k, \tau) \int_{-\infty}^0 d\eta \phi_m(k, \eta) \mathcal{J}(k, \eta, \tau_2). \quad (52)$$

By inspection of Eq. (51) it can be seen that in fact this is no more than the Fourier–Bessel representation of  $\mathcal{J}(\eta, \tau_q)$ . We assume a solution is possible of the form  $\delta G(\tau) = \int_{-\infty}^{\infty} dm \phi_m(\tau) \delta G(m)$  + elements of  $\ker \mathcal{B}_{3/2}$ , where we indicate whether  $\delta G(\tau)$  or some component of its transform  $\delta G(m)$  is under discussion by writing in the argument explicitly. Substituting into Eq. (49) allows one to solve exactly for  $\delta G(\tau)$ :

$$\delta G(k, \tau) = \int_{-\infty}^{\infty} \frac{dm}{-m^2} \phi_m(k, \tau) \int_{-\infty}^0 d\eta \phi_m(k, \eta) \mathcal{J}(k, \eta, \tau_2) - 3 \frac{\delta a}{a} \Big|_{\tau_2} G_0(\tau, \tau_2). \quad (53)$$

The term  $G_0 \in \ker \mathcal{B}_{3/2}$  is chosen to represent the  $\delta$ -function in Eq. (49) because it trivially gives back the same behaviour near  $|\tau| \rightarrow \infty$  as  $G_0$ . We are also writing  $\mathcal{J}(\tau, \tau_2)$  only over the range  $\tau \in \mathbb{R}^-$ , because it is convenient to define the source function in the  $t$ -frame where  $\mathcal{J}(\tau, \tau_2)$  is then undefined for  $\tau > 0$ .

This procedure makes sense provided the Fourier–Bessel transform of  $\mathcal{J}$  exists; to check convergence of the integral, it is necessary to assess the behaviour of  $\mathcal{J}(\eta, \tau_2) \phi_m(\eta)$  both as  $\eta \rightarrow -\infty$  and  $\eta \rightarrow 0$ .

In the  $\eta \rightarrow -\infty$  case, the  $\phi_m(\eta)$  tend to oscillating functions of  $\eta$ . In this case the integral converges provided  $\delta\rho \rightarrow 0$  as  $\eta \rightarrow -\infty$ . The  $\eta \rightarrow 0$  case is more complicated. In this limit,  $\phi_m(\eta) \sim \eta^2$ , whereas  $G_0(\eta) \sim \eta^{-1}$ . The first term in  $\mathcal{J}$  diverges like  $G_0 \delta a/a$ , and since  $\delta a/a \sim \delta\rho/\rho$  this term behaves near  $\eta \rightarrow 0$  like  $a^{-1} \delta\rho$ . To prevent a divergence,  $\delta\rho$  must not diverge faster than  $a$ . Using the thermodynamic redshifting law  $\rho \sim a^{-3(1+w)}$ , where  $p = w\rho$  is the equation of state, this translates into an asymptotic equation of state stiffer than  $w = -4/3$ . This is satisfied for all forms of matter obeying causal propagation [19].

The remaining two terms behave like  $a\delta\rho$ , for which  $\delta\rho$  must go to zero faster than  $a^{-2}$  where the integral would be logarithmically divergent. This gives  $w > -1/3$ , which precludes any form of matter leading to accelerated expansion or quintessence-like behaviour in the late universe, but allows any form of normal matter with  $w \geq 0$ . In particular, if one imagines  $\delta\rho$  dying away in the infinite future (in the  $t$ -frame) as well as the infinite past, then the integral will be well behaved. There are no strong restrictions on the behaviour of  $\delta\rho$  in the intermediate region between the asymptotic past and future.

One can now assemble  $G_0$  and  $\delta G$  to construct the full two-point function, restoring the necessary factors of  $e^{i\mathbf{k} \cdot \mathbf{p}}$  and  $a$ , and integrations over  $\mathbf{k}$ . We let  $\mathbf{x}_1$  approach  $\mathbf{x}_2$ , which gives

$$-iG(\mathbf{x}_1, \mathbf{x}_1) = H_0 \tau \int \frac{d^3 k}{(2\pi)^3} W(k, \tau, \tau_2) \quad (54)$$



where

$$W(k, \tau, \tau_2) = \frac{\pi}{4k} H_0 \tau_2 \left(1 - 3 \frac{\delta a}{a}\right) L^{(1)}(-k\tau_2) L^{(2)}(-k\tau) + i \int_{-\infty}^{\infty} \frac{dm}{m^2} \phi_m(k, \tau) \int_{-\infty}^0 d\eta \phi_m(k, \eta) \mathcal{J}(k, \eta, \tau_2) \quad (55)$$

if  $\tau > \tau_2$  and the same expression with  $H_{3/2}^{(1)}$  and  $H_{3/2}^{(2)}$  interchanged if  $\tau < \tau_2$ .  $L^{(1,2)}(z)$  is defined by Eq. (5) in Section II A, as before. To find the power spectrum one lets  $\tau$  approach  $\tau_2$  and takes a logarithmic derivative with respect to  $k$ . The result is

$$\Delta_\phi^2 = \frac{4\pi k^3}{(2\pi)^3} H_0 \tau W(k, \tau, \tau). \quad (56)$$

To take the  $k \rightarrow 0$  limit, one needs to know the behaviour of  $\mathcal{J}(\eta, \tau)$  at small  $k$ . The term proportional to  $\delta H$  vanishes, because  $\partial G / \partial t$  vanishes on large scales, leaving only the first term which comes from perturbing the Laplacian. As a result,

$$\mathcal{J}(\eta, \tau) \xrightarrow{k \rightarrow 0} H_0 \tau \frac{\pi i k}{2} \frac{\delta a}{a}(\eta) \frac{2^3 \Gamma^2(2/3)}{\pi^2 (-k\eta)(-k\tau)}. \quad (57)$$

Substituting this into the limiting form of Eq. (56) gives

$$\Delta_\phi^2 = \left(\frac{H_0}{2\pi}\right)^2 \times \left(1 - 3 \frac{\delta a}{a} - \frac{2}{\tau} \int_{-\infty}^{\infty} \frac{dm}{m^2} \hat{\phi}_m(\tau) \int_{-\infty}^0 \frac{d\eta}{\eta} \hat{\phi}_m(\eta) \frac{\delta a}{a}(\eta)\right), \quad (58)$$

where we define a new set of functions  $\{\hat{\phi}_m\}$  to be the  $k \rightarrow 0$  limit of the eigenfunctions  $\{\phi_m\}$ . To achieve this result, we have evaluated the answer on the horizon scale  $k = aH = -1/\tau$ .

## V. GRAVITATIONAL FLUCTUATIONS IN A PERTURBED BRANEWORLD

We now aim to repeat this calculation in the braneworld. As before, we consider a de Sitter brane with Hubble parameter  $H_0$  immersed in anti-de Sitter space and allow small fluctuations  $\delta\rho$  in the matter density. However as discussed above these fluctuations are taken to vary with  $\lambda$  in such a way as to keep  $\delta H$  the same. We define this to be our notion of the ‘same’ perturbation in the brane world and in four dimensions.

Firstly, consider some scalar field  $\phi$  propagating over the brane  $\Sigma$ . The operator  $\square$  appearing in the scalar field action is the same as would arise in the four-dimensional case, so the theory is the same as Eq. (42) and the resulting power spectrum satisfies Eq. (58), provided that  $a$  is

taken to satisfy the expansion law for the on-brane cosmological scale factor. One would relabel this quantity  $a_b$  in the brane case.

The case of gravitational waves is not the same. In a general geometry, the graviton wave operator  $\square_{\text{BW}}$  couples the  $t$  and  $y$  dependence of the graviton  $\mathbf{k}$ -modes, so that an explicit solution is extremely difficult. One can always work on the brane universe in black hole coordinates [23, 24], where the metric is explicitly stationary, and one recovers ordinary differential equations. Unfortunately, the boundary conditions are non-trivial to apply. In this section we make progress by a different route.

We begin by rewriting the general formula Eq. (21) for  $n(t, y)$  in terms of  $H'$ , where

$$\dot{H} = \dot{\phi} H' = -\frac{2\mu}{\kappa_4^2} \frac{H'^2}{\sqrt{H^2 + \mu^2}}. \quad (59)$$

Since  $\dot{H} \propto H'^2$ , if we perturb around the de Sitter solution, the term  $H' = \delta H$  squares to zero. Hence, for a perturbed de Sitter brane, we still retain  $n = a/a_b$ . We emphasize that this is only true for perturbations around de Sitter space supported by a scalar field where the background  $H$  satisfies  $H' = 0$ . When calculating spectral indices we will again endow  $H$  with very weak time dependence, but for the purposes of the calculation presented in this section the background  $H$  is to be regarded as fixed, in analogy with the four dimensional calculation of  $\Delta_\phi^2$ .

Let us write the metric functions  $a$  and  $n$  as in Eq. (20)–(21),

$$n^2(y) = \frac{H^2}{2\mu^2} [\cosh 2\mu(y_h - y) - 1] \quad (60)$$

and  $a(t, y) = a_b(t)n(y)$  where  $a_b(t)$  is the scale factor on the brane. Under a variation  $H \mapsto H_0 + \delta H$ , the function  $n(t, y)$  becomes

$$n(t, y) \mapsto \left(1 + \frac{\delta H}{H_0}\right) \frac{H_0}{\sqrt{2}\mu} [\cosh 2\mu(y_h + \delta y_h - y) - 1]^{1/2}, \quad (61)$$

since the horizon location  $y_h$  in principle depends on time. However, the variation in  $y_h$  can be safely ignored, because the small term  $\delta y_h$  vanishes inside cosh, that is,  $\cosh(X_0 + \varepsilon) = \cosh X_0 + O(\varepsilon^2)$  for any  $X_0$  and small quantity  $\varepsilon$ . Therefore, one may still take the effective horizon to sit at  $y = y_h$  and write

$$n(y) \mapsto (1 + \tilde{\delta}) n_0(y), \quad \tilde{\delta} = \frac{\delta H}{H_0} \quad (62)$$

where  $n_0(y)$  is the unperturbed function Eq. (60). The metric is  $ds_5^2 = n_0^2 ds_4^2 + dy^2$ , where

$$ds_4^2 = (1 + \tilde{\delta})^2 [-dt^2 + (a_b + \delta a_b)^2 \delta_{ij} dx^i dx^j]. \quad (63)$$

This guarantees that  $\square_{\text{KG}}$  and  $\square_{\text{Y}}$  separate, and under these circumstances  $\square_{\text{Y}}$  depends only on  $y$ . The opera-

tors  $\square_{\text{KG}}$  and  $\square_Y$  take the explicit form

$$\begin{aligned}\square_{\text{KG}} &= -\frac{\partial^2}{\partial t^2} - \left(3H + 2\frac{\partial\tilde{\delta}}{\partial t}\right) \frac{\partial}{\partial t} + \frac{\Delta}{a_b^2}, \\ \square_Y &= \frac{\partial^2}{\partial y^2} + 4\frac{n'_0}{n_0} \frac{\partial}{\partial y}.\end{aligned}\quad (64)$$

The operator  $\square_Y$  is again an honest Sturm–Liouville operator. We consider weighted eigenfunctions of the form

$$\square_Y \mathcal{E}_\alpha = -(\alpha^2/n_0^2) \mathcal{E}_\alpha. \quad (65)$$

The  $\mathcal{E}_\alpha$  are chosen by fixing the derivatives  $\mathcal{E}'_\alpha$  to vanish at  $y = 0$  and  $y = y_h$ ; with this choice, they can be made orthogonal in the inner product  $(\mathcal{E}_\alpha | \mathcal{E}_\beta) = \int d\mu(y) \mathcal{E}_\alpha^* \mathcal{E}_\beta$  as before, where  $d\mu(y) = n_0^2 dy$ . In particular, the normalization  $F$  of the  $\mathcal{E}_0$  eigenfunctions depends only on  $H_0$  and not  $\delta H$ , because  $\square_Y$  does not see the perturbations: it is the same as the unperturbed operator. The gravitational action, after decomposing into  $SO(3)$  polarizations, is equivalent to a certain number of copies of the scalar field action, one for each polarization state of the graviton. The action is (cf. Eq. (30))

$$\frac{1}{2} \int_\Sigma dx \sum_\alpha E^\alpha \left( \frac{\square_{\text{KG}}}{(1+\tilde{\delta})^2} - \alpha^2 \right) E^\alpha \quad (66)$$

where the integration is in the metric  $ds_4^2$  and all functions  $n_0(y)$  have disappeared.

The important feature here is that Eq. (66) is not the same as the action for a perturbed four-dimensional scalar field. Some structure left over from the higher-dimensional graviton operator  $(\square_{\text{KG}}/n^2) + \square_Y$  is still visible, which modifies the result.

Consider the massless  $\alpha = 0$  mode. Just as in the unperturbed case, this is the important contribution for perturbations generated during inflation. The higher Kaluza–Klein modes with  $\alpha > 3H/2$  are heavy in the sense that their fluctuations are not amplified, so they may be discarded to a good approximation in this analysis. The Green’s function  $G$  satisfies

$$\frac{1}{(1+\tilde{\delta})^2} \square_{\text{KG}} G(x_1, x_2) = \delta_4(x_1 - x_2) \quad (67)$$

with  $\delta_4$  the delta function in  $ds_4^2$ . As before we seek to solve  $G$  as the Fourier transform  $G(\mathbf{k}, \tau, \tau_2)$  of a perturbation expansion  $G = G_0 + \delta G$  in  $\delta H$ . We change to conformal time  $\tau$  and write  $G_0 \rightarrow a_b G_0$ . The  $O(1)$  equation is  $\mathcal{B}_{3/2} G_0 = \delta(t - t_2)/a_0$ . (When writing  $a_b$ , we always mean the unperturbed  $a_b$  which satisfies  $a_b = e^{H_0 t} = -(H_0 \tau)^{-1}$ , with  $\tau$  the unperturbed conformal time. Variations in  $a_b$  are explicitly written as  $\delta a_b$ .) This is the same as Eq. (44) for the unperturbed four-dimensional Green’s function, and so shares the same solution.

The  $O(\delta H)$  equation differs from the four-dimensional case Eq. (49). It is

$$\begin{aligned}\mathcal{B}_{3/2} \delta G &= \left[ 2k^2 G \frac{\delta a_b}{a_b} - \left( 3\delta H + 2\dot{\tilde{\delta}} \right) \left( a_b \frac{\partial G_0}{\partial \tau} - a^2 H_0 G_0 \right) \right] \\ &\quad - \left( 3 \frac{\delta a_b}{a_b} + 2\tilde{\delta} \right) \frac{\delta(\tau - \tau_2)}{a_b} \\ &= \tilde{\mathcal{J}}(\tau, \tau_2) - \left( 3 \frac{\delta a_b}{a_b} + 2\tilde{\delta} \right) \frac{\delta(\tau - \tau_2)}{a_b}.\end{aligned}\quad (68)$$

Nonetheless, this expression has exactly the same structure as the four-dimensional problem: the operator on the left-hand side is  $\mathcal{B}_{3/2}$ , which permits a solution for  $\delta G$  as a transform in the eigenfunctions of  $\mathcal{B}_{3/2}$ . The right-hand side takes the form of some source term  $\tilde{\mathcal{J}}$  and a contribution proportional to  $\delta(\tau - \tau_2)$ . The difference lies in the explicit form of  $\tilde{\mathcal{J}}$  and the coefficient of the  $\delta$ -distribution.

The solution of Eq. (68) therefore comes out to be

$$\begin{aligned}\delta G &= \int_{-\infty}^{\infty} \frac{dm}{-m^2} \phi_m(\tau) \int_{-\infty}^0 d\eta \tilde{\mathcal{J}}(\eta, \tau_2) \phi_m(\eta) \\ &\quad - \left( 3 \frac{\delta a_b}{a_b} + 2\delta g \right) G_0(\tau, \tau_2).\end{aligned}\quad (69)$$

The conditions for the  $\mathcal{B}_{3/2}$  or Fourier–Bessel transform to make sense are the same as the four-dimensional case; in particular,  $\delta\rho$  must vanish as  $\eta \rightarrow -\infty$  and obey an equation of state  $p = w\rho$  with  $w > -1/3$  as  $\eta \rightarrow 0$ . There is very little loss of generality in assuming that  $\delta\rho$  dies away in the asymptotic future as well as the asymptotic past.

One now writes the full two-point function in a manner analogous to Eq. (54) and takes the limit  $x_1 \rightarrow x_2$  to give a power spectrum. This part of the argument involves manipulations very similar to those leading to Eq. (58), so we do not write them out explicitly. One then takes the large scale limit  $k \rightarrow 0$ . The source function  $\tilde{\mathcal{J}}$  obeys similar asymptotics to Eq. (57): in particular, terms arising from  $\partial G_0 / \partial t$  vanish, so the terms involving  $\delta H$  and  $\dot{\tilde{\delta}}$  disappear. Therefore, one has

$$\begin{aligned}\Delta_{T,5}^2 &= 8\kappa_4^2 F^2 \left( \frac{H_0}{2\pi} \right)^2 \left( 1 - 3 \frac{\delta a_b}{a_b} - 2\tilde{\delta} \right. \\ &\quad \left. - 2 \int_{-\infty}^{\infty} \frac{dm}{m^2} \frac{\hat{\phi}_m(\tau)}{\tau} \int_{-\infty}^0 d\eta \frac{\hat{\phi}_m(\eta)}{\eta} \frac{\delta a_b}{a_b} \right)\end{aligned}\quad (70)$$

This is the power spectrum, as seen on the brane, of gravitational waves excited during the perturbed inflationary epoch.

Eq. (70) has the important property, alluded to in the introduction, that as  $\lambda \rightarrow \infty$  it does not go over to the corresponding four-dimensional form  $\Delta_{T,4}^2$ . This is because the contribution  $\tilde{\delta} = \delta H/H$  does not vanish as  $\lambda \rightarrow \infty$ .

## VI. CONSISTENCY RELATIONS ON PERTURBED BACKGROUNDS

One does not expect that a relationship between observable quantities of the form  $\Delta_T^2/\Delta_\zeta^2 \propto n_T$  should hold for either of these perturbed universes. In any case, this relation is only true to first order in the slow-roll expansion and is known to receive corrections of a different functional form at next-order [9]. For this reason, it is clear that the consistency relation as we have written it here, as direct proportionality between the ratio of  $\Delta_T^2$  and  $\Delta_\zeta^2$  and the tensor spectral index  $n_T$ , is only an approximation to whatever exact connexion exists between  $\Delta_T^2$ ,  $\Delta_\zeta^2$ ,  $n_T$  and  $n_\zeta$ . In particular, the next-order result involves all four observable quantities. However one can meaningfully ask about the manner in which this approximate relationship is broken by introducing a perturbation  $\delta H$ .

For the purposes of this section, it is useful to recast the results Eq. (58) and Eq. (70) for the power spectra  $\Delta_{T,4}^2$  and  $\Delta_{T,5}^2$  in a slightly different form, by making the prefactor involve the full perturbed  $H$  rather than its fixed background value  $H_0$ . This corresponds simply to including a contribution  $-2\delta H/H_0$  in the  $O(\delta H)$  terms. In fact, for the argument we are about to make the detailed form of these terms is not important so we will denote them collectively just by  $\gamma$ . Then, comparing Eq. (58) and Eq. (70),

$$\Delta_{T,4}^2 = \left(\frac{H}{2\pi}\right)^2 (1 + \gamma) \quad (71)$$

and

$$\Delta_{T,5}^2 = \left(\frac{H}{2\pi}\right)^2 (1 + \gamma - 2\tilde{\delta}). \quad (72)$$

Let us assume that the scalar field  $\phi$  dominates the energy density of the universe. The matter power spectrum still [21] satisfies  $\Delta_\zeta^2 = (H/\dot{\phi})^2 \Delta_\phi^2$  in the four-dimensional and brane case equally, where  $\Delta_\phi^2$  is given by Eq. (58). Here, we assume that  $H$  includes corrections owing to the perturbation  $\delta\rho$  in the density, which might be sourced by some perturbation  $\delta\phi$  in the scalar field. However, we will continue to assume that the four- and five-dimensional Hamilton–Jacobi equations Eq. (35) and Eq. (38) hold, with  $H$  including contributions of  $O(\delta\rho)$ . The tensor power spectrum in the four-dimensional case is given by an identical expression to its unperturbed counterpart, namely  $\Delta_{T,4}^2 = 8\kappa_4^2 \Delta_\phi^2$ . In the brane world, one has instead

$$\Delta_{T,5}^2 = 8\kappa_4^2 F^2 \Delta_\phi^2 (1 - 2\tilde{\delta}), \quad (73)$$

as can be seen by inspection of Eq. (71) and Eq. (72).

Then the ratio  $\Delta_T^2/\Delta_\zeta^2$  takes the following form:

$$\begin{aligned} \frac{\Delta_{T,4}^2}{\Delta_{S,4}^2} &= \frac{32 H'^2}{\kappa_4^2 H^2}, \\ \frac{\Delta_{T,5}^2}{\Delta_{S,5}^2} &= \frac{32\mu^2}{\kappa_4^2} F^2 \frac{H'^2}{H^2} \frac{1}{H^2 + \mu^2} (1 - 2\tilde{\delta}). \end{aligned} \quad (74)$$

As noted above, we are assuming that  $\dot{\phi}$  and  $H$  are related by perturbed versions of the Hamilton–Jacobi equations, and  $H$  should be expanded as  $H = H_0 + \delta H$ . This procedure is rather approximate, but one is already obliged to introduce approximations into the calculation when differentiating with respect to  $\log k$ , and controlling each of these estimates is not trivial. As discussed above, neither of these ratios are now equal to their respective tensor indices  $n_T$ .

Let us deal with the four dimensional case first. In any event, we need an expression connecting observable quantities in four dimensions with which to compare any such connexion in five dimensions. In fact, it is no longer easy to write down any connexion between the four principal observables  $\Delta_{T,4}^2$ ,  $\Delta_{\zeta,4}^2$ ,  $n_{T,4}$  and  $n_{\zeta,4}$ . By differentiating Eq. (71) and the expression  $\Delta_{\zeta,4}^2 = (\dot{\phi}/H)^2 \Delta_\phi^2$  for  $\Delta_{\zeta,4}^2$  one can find the spectral indices. They are,

$$n_{T,4} = -\frac{4}{\kappa_4^2} \left(\frac{H'}{H}\right)^2 + \frac{d\gamma}{d\log k}, \quad (75)$$

and

$$n_{\zeta,4} - 1 = -\frac{8}{\kappa_4^2} \left(\frac{H'}{H}\right)^2 + \frac{4}{\kappa_4^2} \frac{H''}{H} + \frac{d\gamma}{d\log k}. \quad (76)$$

All of these expressions are assumed to be taken at first order in the slow-roll expansion, and first order in  $\gamma = O(\delta H)$ . Where products of two or more slow-roll parameters appear, or a product of a quantity at  $O(\delta H)$  and a slow-roll parameter, they are assumed to be discarded together with contributions of  $O(\delta H^2)$ . To express  $n_{T,4}$  in terms of the other observables, one must find some way to eliminate  $d\gamma/d\log k$ . This quantity looks like a contribution of order (slow roll) $^{1/2} \times O(\delta H)$ , so we do not discard it. (Because of this, one can regard our result as a kind of  $1\frac{1}{2}$ -order expansion.) Since  $\gamma$  involves a particular Fourier–Bessel transform of the quite general source function  $\mathcal{J}$  (or  $\tilde{\mathcal{J}}$ ), which does not appear in any other quantity such as  $H$ , it would appear that the only way of eliminating it is to introduce  $n_{\zeta,4}$ . The terms involving  $H'/H$  can be rewritten using Eq. (74), but there is no way of rewriting  $H''/H$  using only the observables under discussion.

To circumvent this difficulty requires the introduction of extra observable parameters with which to broaden the range of quantities we can express using them. In the present case, it is most convenient to introduce the running  $r_{T,4}$  of the tensor spectral index which was defined in Eq. (10). This satisfies

$$r_{T,4} = -8 \frac{\Delta_{T,4}^2}{\Delta_{\zeta,4}^2} \left(\frac{H''}{H} - \frac{H'^2}{H^2}\right) + \frac{d^2\gamma}{d[\log k]^2}. \quad (77)$$

The second derivative of  $\gamma$  is of order an  $O(\delta H)$  term multiplied by a slow-roll parameter, so it can be discarded at the order to which we are working. Combining Eq. (77) with Eqs. (75)–(76) allows us to write an expression for  $n_{T,4}$  involving only quantities which are in principle observable:

$$n_{T,4} \approx \frac{3}{16} \frac{\Delta_{T,4}^2}{\Delta_{\zeta,4}^2} + (n_{\zeta,4} - 1) + \frac{r_{T,4}}{2\kappa_4^2} \frac{\Delta_{\zeta,4}^2}{\Delta_{T,4}^2}. \quad (78)$$

We write an approximate equality  $\approx$  to indicate that this result is true only to first order in a combined slow-roll/ $O(\delta H)$  expansion.

The principal result of this paper is that no such comparable relation between observational quantities exists in the brane world. One can again obtain spectral indices using the usual argument; for example, the tensor spectral index satisfies

$$n_{T,5} = -\frac{4\mu^2}{\kappa_4^2} F^2 \left( \frac{H'}{H} \right)^2 \frac{1}{H^2 + \mu^2} + \frac{d\gamma}{d \log k} - 2 \frac{d\tilde{d}}{d \log k}. \quad (79)$$

This involves an extra  $O(\delta H)$  correction  $d\tilde{d}/d \log k$ . One can write similar expressions for  $n_{\zeta,5}$ ,  $r_{T,5}$  and so on which we shall suppress because of their complexity. However, none of these quantities contains any counter-term with which one could balance the  $d\tilde{d}/d \log k$  appearing in Eq. (79). This is because no such contribution occurs in the  $\zeta$  quantities, and higher derivatives of  $n_{T,5}$  will not contain  $\delta H$  at first order: as in four dimensions, second derivatives of  $\gamma$  and  $\delta H$  are at second order in the slow-roll/ $O(\delta H)$  expansion. Thus it is quite impossible to write any relationship between observable quantities which respects Eq. (79).

This is a rather stronger statement than that the four-dimensional consistency relation no longer holds in the brane world: rather, we have shown that there is no consistency relation at all, involving  $n_T$ , that holds in this universe. This means that all four observables  $n_\zeta$ ,  $n_T$ ,  $\Delta_\zeta^2$  and  $\Delta_T^2$  as well as any running in them are independent. We consider this as a kind of manifestation of the well-known result that the four-dimensional cosmological perturbation theory does not constitute a closed system: one needs extra information about the behaviour of the bulk. The change in the behaviour of gravitational quantities is a result of an off-brane effect which cannot be reconstructed in terms of purely brane-based observables.

Although we have exhibited this property only in a single example model, we expect such behaviour to be quite generic. We would like to stress that not only can one not write a consistency relation for the brane world at finite  $\lambda$ , no such consistency relation exists as  $\lambda \rightarrow \infty$  either. This is because the  $\delta H/H$  contribution which gives trouble does not disappear in this limit; it is a consequence of the fact that  $\Delta_{T,5}^2$  does not approach  $\Delta_{T,4}^2$  as  $\lambda \rightarrow \infty$ .

## VII. CONCLUSIONS

The brane universe offers the prospect of reproducing four-dimensional physics from string vacua without the necessity of compactifying extra dimensions on Planck-sized manifolds. One would like to investigate the possible consequences of scenarios of this type: apart from signatures in the particle physics sector, the other major high energy laboratory in which one could conceive of observing such consequences is the early universe. So it is important to study the implications of early universe cosmology in the brane world scenario.

In this paper, we have developed a perturbation expansion for the gravitational wave modes around the pure de Sitter case  $H = \text{constant}$ . This perturbation series can be used in the brane world and in four dimensional equally. We use this technology to calculate the power spectrum of scalars and gravitational waves as seen on the brane, or in four dimensions, and write a consistency relation in the four-dimensional case. We also show that no such consistency relation exists in the brane world.

This analysis confronts a troubling feature of the brane world model: it predicts an identical observational degeneracy in comparison with the conventional four-dimensional cosmology. We have shown, by an explicit calculation, that degeneracies of this type are not generic. If one takes a four-dimensional cosmology exhibiting some degeneracy, then one should not expect, in general, for this degeneracy to remain when one the four-dimensional cosmology as a brane. This result is important; a complete degeneracy would hinder any attempt to observationally reconstruct the inflaton potential.

Our calculation relies on exploiting a technical device to calculate the tensor power spectrum in a model perturbed around a de Sitter brane carrying a single scalar field. This extends the range of models in which one knows how to solve for the spectrum of gravitational waves produced during an inflationary epoch. This is a hard problem, whose complete solution is not yet understood (but see Ref. [32] for a calculation of the tensor spectrum in the case where the brane carries a large  $N$  CFT, using the AdS/CFT correspondence). Our method will not easily generalize to full case of arbitrary time evolution on the brane, but may suggest future directions in which to proceed. One such possibility is to study the brane universe in explicitly static Schwarzschild–Anti de Sitter (SAdS) coordinates, where there is a holonomic timelike Killing vector  $\partial/\partial T$ . The graviton field equation is then independent of  $T$  and becomes an ordinary differential equation, similar to the Regge–Wheeler equation of black hole perturbation theory. The brane appears as a Neumann boundary condition applied to what is effectively a moving mirror, and it is possible that this framework is accessible to analytic attack. Our calculation does not yet include back reaction from other fields on the brane, so it not general enough (for example) to include other types of matter, or to generalize to a second order result.

We have argued that one can associate the persistence of a consistency relation with the property that  $\Delta_{T,5}^2$  (and therefore  $n_{T,5}$ ) goes over to its four-dimensional counterpart as one decouples the brane from the bulk by taking  $\lambda \rightarrow \infty$ . This property holds for the exact de Sitter imbedding but need not hold for more general cosmologies. The central feature of this correspondence is that the relationship between observables on the brane is independent of the tension  $\lambda$ , and so is guaranteed to match the four-dimensional result. This property is enforced by the differential equation (40). We exhibit directly a solution corresponding to a marginally perturbed de Sitter geometry for which  $\Delta_{T,5}^2$  does not approach its four-dimensional counterpart in the decoupling limit, a consistency relation does not exist, and the relationship between observables is not  $\lambda$ -independent.

One might worry that our analysis involves a small perturbation propagating on top of an inflating universe. By the usual inflationary arguments this perturbation should decay exponentially quickly as inflation proceeds, and become negligible. Although this is true, we conceive of the perturbation as a probe of gravity's behaviour in the two cosmologies, regardless of its absolute magnitude. Because we did not specify how the perturbation to the four-dimensional cosmology carried by the brane was to be sourced, one can possibly imagine it to derive from bulk effects, such as an impinging gravitational wave. The comparison four-dimensional calculation should then be understood as a formal device. Alternatively, it is simple an imbedded cosmological model in some open neighbourhood of the exact de Sitter solution. However one interprets the calculation, however, the underlying principle is the same: one does not recover four-dimensional quantities after taking  $\lambda$  to infinity.

### Acknowledgments

DS is supported by a PPARC Studentship. We would like to thank David Wands for useful conversations regarding this work.

### APPENDIX A: THE NORMALIZATION FUNCTION $F$

In this section we sketch how the normalization function  $F$  and the central differential equation Eq. (40) are obtained.

One defines  $F$  to satisfy  $2\mu F^2 = \mathcal{E}_0^2$ , where  $\mathcal{E}_0$

is the zero-mode of  $\square_Y$ . The  $\mathcal{E}_\alpha$  are normalized in the Sturm–Liouville measure arising from  $\square_Y$ , that is,  $2 \int dy n^2 \mathcal{E}_\alpha \mathcal{E}_\beta = \delta_{\alpha\beta}$ . The factor of 2 has been added to take account of the other branch of the orbifold, since we work on  $y \in [0, y_n]$ . Because the  $\mathcal{E}_0$  are independent of  $y$ , this just says  $2\mu F^2 \int n^2 dy = 1$ .

It is easy to evaluate this integral directly, but for the purposes of obtaining Eq. (40), it is convenient to employ the relation  $n' = -\sqrt{H^2 + \mu^2 n^2}$  which arises from the Einstein field equation. In that case, the normalization requirement depends only on the integral of the purely geometrical quantity  $n$ ,

$$2\mu F^2 \int_0^1 \frac{n^2 dn}{\sqrt{H^2 + \mu^2 n^2}} = 1. \quad (\text{A1})$$

This does not depend on a detailed knowledge of the form of  $n$ , except through  $n'$ . Here  $\mu$  is the ratio  $\kappa_4^2/\kappa_5^2$  of the four- and five-dimensional gravitational couplings and  $\ell$  is the AdS radius, which in the case of vanishing four-dimensional cosmological constant equals  $\mu$ . This is the case throughout the main body of this paper. One now makes a trigonometric substitution to evaluate the integral. The result is

$$\frac{\mu}{\ell} F^2 \left( \sqrt{1 + \frac{H^2}{\ell^2}} - \frac{H^2}{\ell^2} \text{arc sinh} \frac{\ell}{H} \right) = 1. \quad (\text{A2})$$

If  $\mu = \ell$  then this result agrees with Refs. [15, 30]. To derive Eq. (40), one can differentiate this result directly, but it is easier to proceed as follows. Multiply Eq. (A1) by  $H^2$  and differentiate logarithmically. One finds,

$$\frac{d \log HF}{d \log H} + \mu F^2 \frac{d}{d \log H} \int_0^1 \frac{n^2 dn}{\sqrt{H^2 + \mu^2 n^2}} = 1. \quad (\text{A3})$$

It is now easy to differentiate under the integral sign, and integrate the resulting expression. That gives

$$\frac{d}{d \log H} \int_0^1 \frac{n^2 dn}{\sqrt{H^2 + \mu^2 n^2}} = \frac{1}{\mu F^2} - \frac{1}{\sqrt{H^2 + \mu^2}}. \quad (\text{A4})$$

Substituting this into Eq. (A3) gives the result Eq. (40). This relation was first noticed by the authors of Ref. [16].

Because the normalization integral does not involve integrating over a solution to the field equation  $\square\phi = 0$ , one can interpret this result as a statement about the dependence of the metric  $g_{ab}$  on the initial conditions prescribed on the brane.

- 
- [1] D. Spergel, L. Verde, H. Peiris, E. Komatsu, M. Nolta, C. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, et al., *Ap. J.* (2003), astro-ph/0302209.  
[2] A. Starobinsky, *Phys. Lett. B* **91**, 99 (1980).

- [3] E. Halyo (2002), hep-th/0203235.  
[4] A.H. Guth, *Phys. Rev. D* **23**, 347 (1981).  
[5] V. Mukhanov, H. Feldman, and R. Brandenberger, *Phys. Rep.* **215**, 203 (1992).

- [6] E. Hawkins, S. Maddox, S. Cole, D. Madgwick, P. Norberg, J. Peacock, I. Baldry, C. Baugh, J. Bland-Hawthorn, T. Bridges, et al., MNRAS (2002), astro-ph/0212375.
- [7] A. Starobinsky, Sov. Astr. Lett. **11**, 133 (1985).
- [8] A. Liddle and D. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge, 2000).
- [9] J. Lidsey, A. Liddle, E. Kolb, E. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys **69**, 373 (1997), astro-ph/9508078.
- [10] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-ph/9905221.
- [11] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.
- [12] P. Hořava and E. Witten, Nucl. Phys. B **460**, 506 (1996), hep-th/9510209.
- [13] P. Hořava and E. Witten, Nucl. Phys. B **475**, 94 (1996), hep-th/9603142.
- [14] R. Maartens, D. Wands, B.A. Bassett, and I.P.C. Heard, Phys. Rev. D **62**, 041301 (2000), hep-ph/9912464.
- [15] D. Langlois, R. Maartens, and D. Wands, Phys. Lett. B **489**, 259 (2000), hep-th/0006007.
- [16] G. Huey and J. Lidsey, Phys. Lett. B **514**, 217 (2001), astro-ph/0104006.
- [17] G. Huey and J. Lidsey (2002), astro-ph/0205236.
- [18] A.R. Liddle and A.N. Taylor, Phys. Rev. D **65**, 041301 (2002), astro-ph/0109412.
- [19] S. Hawking and G. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973).
- [20] N. Birrell and P. Davies, *Quantum fields in curved space* (Cambridge University Press, Cambridge, 1982).
- [21] D. Wands, K.A. Malik, D.H. Lyth, and A.R. Liddle, Phys. Rev. D **62**, 043527 (2000), astro-ph/0003278.
- [22] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1994).
- [23] P. Bowcock, C. Charmousis, and R. Gregory, Class. Quant. Grav. **17**, 4745 (2000), hep-th/0007177.
- [24] S. Mukohyama, T. Shiromizu, and K.I. Maeda, Phys. Rev. D **62**, 024028 (2000), hep-th/9912287.
- [25] S.S. Gubser, Phys. Rev. D **63**, 084017 (2001), hep-th/9912001.
- [26] H. Verlinde, Nucl. Phys. B **580**, 264 (2000), hep-th/9906182.
- [27] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. B **565**, 269 (2000), hep-th/9905012.
- [28] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B **477**, 285 (2000), hep-th/9910219.
- [29] T. Shiromizu, K.I. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000), gr-qc/9910076.
- [30] D. Gorbunov, V. Rubakov, and S. Sibiryakov, JHEP **015**, 0110 (2001), hep-th/0108017.
- [31] G. Giudice, E. Kolb, J. Lesgourgues, and A. Riotto (2002), hep-ph/0207145.
- [32] S.W. Hawking, T. Hertog, and H.S. Reall, Phys. Rev. D **62**, 043501 (2000), hep-th/0003052.
- [33] A. Novikov and L. Kofman (2002), hep-th/0209133.
- [34] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, 1953).